06-262: Review Lecture Notes

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DISCLAIMER: This material does not contain all topics covered in the course. It also does not only include what will be in the final exam.

1 Linear Algebra

1.1 Determinants of Matrices

The determinant is a value associated with a square matrix A. Can be calculated via the Laplace expansion (or cofactor expansion). If A is n-by-n, then:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} \tag{1}$$

where $M_{i,j}$ is the (i, j) minor of A. Alternative method: Cramer's rule.

MATLAB function to calculate the determinant of a matrix: `det`.

1.2 Main Problem

Solve a system of linear equations

$$Ax = b \tag{2}$$

where A is a matrix of coefficients, x is a vector of variables (unknowns), and b is a vector of right-hand side (RHS) values.

For a square system, i.e., A is n-by-n, a unique solution exists if A is non-singular. In other words, $\det(A) \neq 0$. If A is singular, then it is not invertible. Otherwise, the solution is given by

$$x = A^{-1}b \tag{3}$$

In practice, however, one would *not* multiply the explicit inverse of A by b. Use Gaussian Elimination with (partial) pivoting, decomposition methods (LU, QR), or iterative methods (Gauss-Seidel).

MATLAB function to solve system of linear equations: `mldivide` (backslash operator `\`).

1.3 Eigenvalues and Eigenvectors

Interpretation: Given a square matrix A. Almost all vectors change direction when multiplied by A. Certain exceptional vectors v (eigenvectors) are in the same direction as Av. Multiply an eigenvector v by A, and the vector Av is a number λ (eigenvalue) times the original v. Mathematically:

$$Av = \lambda v \tag{4}$$

We say that λ is an eigenvalue of A and v is an eigenvector associated with eigenvalue λ.

For example, the reflection matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Therefore, when multiplied by A, v_1 remains unchanged, but v_2 is reversed (signs are reversed) as shown in Figure 1.

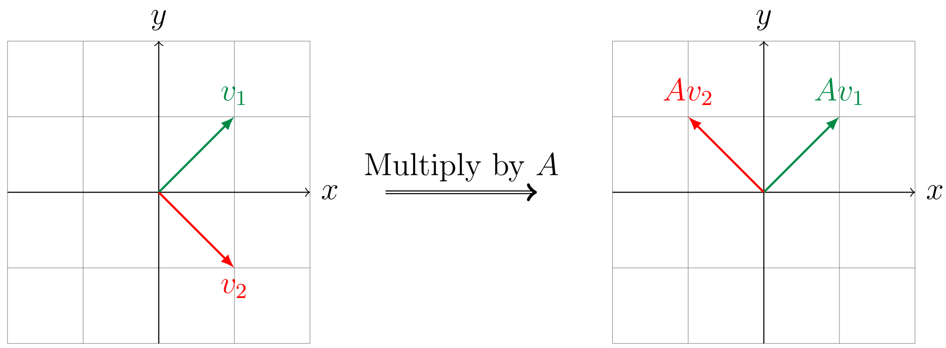

Figure 1: Illustration of a reflection matrix A and its eigenvectors v_1 and v_2.

Calculate eigenvalues by finding the roots of the characteristic polynomial:

$$\det(A - \lambda I) = 0 \tag{5}$$

The corresponding eigenvectors are calculated either by the application of equation (4) or equivalently:

$$(A - \lambda I)v = 0 \tag{6}$$

MATLAB function to compute eigenvalues and eigenvectors: `eig`.

2 Ordinary Differential Equations (ODEs)

Solution of an algebraic equation: scalars.
Solution of a differential equation: functions.

2.1 First-Order ODEs

Consider the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + a(x)y = b(x) \tag{7}$$

where y and x are the dependent and independent variables, respectively, and $a(\cdot)$ and $b(\cdot)$ are (possibly) functions of x.

The integrating factor of the ODE in equation (7) is

$$\mu(x) = \exp\left(\int a(x)\mathrm{d}x\right) \tag{8}$$

The general solution is given by

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)b(x)\mathrm{d}x + C\right] \tag{9}$$

where C is a constant of integration, which can be determined given an initial condition.

2.2 Second-Order Linear ODEs

Consider the ODE

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x) \tag{10}$$

where a, b, and c are scalars, and $f(\cdot)$ is (possibly) a function of x.

The general solution of the ODE in equation (10) contains two parts: complementary ($y_c(x)$) and particular ($y_p(x)$) solutions. Thus

$$y(x) = y_c(x) + y_p(x) \tag{11}$$

The complementary solution is the solution to the homogeneous ODE associated with the original ODE defined in equation (10):

$$a\frac{\mathrm{d}^2y_c}{\mathrm{d}x^2} + b\frac{\mathrm{d}y_c}{\mathrm{d}x} + cy_c = 0 \tag{12}$$

To solve it, obtain the roots of the characteristic (auxiliary) equation (polynomial):

$$a\lambda^2 + b\lambda + c = 0 \tag{13}$$

Three cases are possible (C_1 and C_2 are constants of integration):

Case 1: Real and distinct roots λ_1 and λ_2:

$$y_c(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$

Case 2: Complex conjugate roots $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$:

$$y_c(x) = e^{\alpha x}(C_1 \sin\beta x + C_2 \cos\beta x)$$

Case 3: Real and equal roots $\lambda_1 = \lambda_2 = \lambda$:

$$y_c(x) = (C_1 + C_2x)e^{\lambda x}$$

The particular solution can be obtained via the Method of Undetermined Coefficients (for specific forms of $f(\cdot)$) or the Method of Variation of Parameters (much more general, but requires knowing the complementary solution and integrating functions).

2.3 Numerical Methods

Consider the first-order Initial Value Problem (IVP)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \quad y(a) = y_a, \quad x \in [a, b] \tag{14}$$

Main idea: discretize the domain (independent variable) and predict the dependent variable for each point in the grid.

We consider fixed step size $h \equiv \Delta x$. If the domain is $a \leq x \leq b$, we have

$$x_{n+1} = x_n + h = a + nh, \quad n = 0, 1, \ldots, N \tag{15}$$

where $N = \frac{b-a}{h}$. For each x_n in the grid the numerical method predicts the associated y_n. Figure 2 illustrates the procedure.

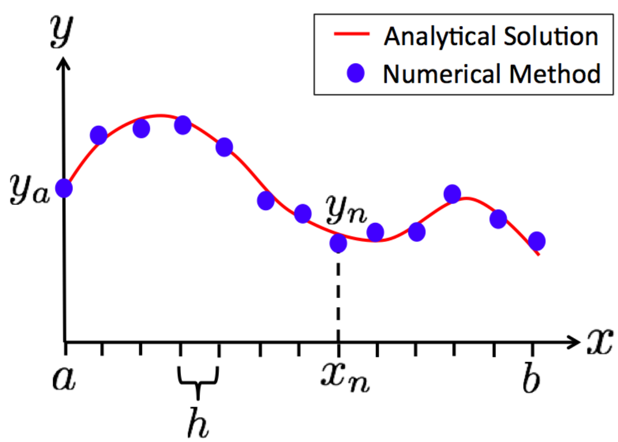

Figure 2: Illustration of numerical method with fixed step size to solve ODEs.

Euler's method is a simple numerical based on the forward finite difference approximation of a first-order derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx \frac{y_{n+1} - y_n}{h} \tag{16}$$

Substituting the RHS of the above expression into equation (14) yields

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, \ldots, N \tag{17}$$

with $x_0 = a$ and $y_0 = y_a$.

Runge-Kutta fourth-order method is more accurate as it considers the weighted average of four increments

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad n = 0, 1, \ldots, N \tag{18}$$

where

$$
\begin{aligned}
k_1 &= f(x_n, y_n) \\
k_2 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{h}{2}k_1\right) \\
k_3 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{h}{2}k_2\right) \\
k_4 &= f(x_n + h, y_n + hk_3)
\end{aligned}
$$

Consider the second-order Boundary Value Problem (BVP)

$$
a(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}y}{\mathrm{d}x} + c(x)y = f(x), \quad y(x_1) = y_1,\ y(x_2) = y_2, \quad x \in [x_1, x_2] \tag{19}
$$

The derivatives can be approximated with central finite difference formulas as follows:

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &\approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &\approx \frac{y_{n+1} - y_{n-1}}{2h}
\end{aligned}
$$

Substituting the expressions above into equation (19) yields

$$
a(x_n)\left(\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}\right) + b(x_n)\left(\frac{y_{n+1} - y_{n-1}}{2h}\right) + c(x_n)y_n = f(x_n), \quad n = 1, 2, \ldots, N \tag{20}
$$

and the boundary conditions

$$
y_0 = y_1 \tag{21}
$$

$$
y_{N+1} = y_2 \tag{22}
$$

These equations form a tridiagonal matrix system of equations.

2.4 MATLAB Functions

Analytical solution of ODEs: `dsolve`.
Numerical solution of ODEs: `ode45`, `bvp4c`, etc.

3 Laplace Transform

3.1 Definition

The Laplace Transform (LT) is usually employed in problems where time is the independent variable. It changes the domain of functions from time t to frequency s. The LT of a function $f(t)$ is defined by

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{23}
$$

The Inverse Laplace Transform (ILT) is usually not discussed in details in undergraduate courses, but it is defined as follows:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} F(s) e^{st} \mathrm{d}s \tag{24}$$

In practice, we mainly need to apply tabulated expressions of $\mathcal{L}\{f(t)\}$ and $\mathcal{L}^{-1}\{F(s)\}$.

3.2 Application to the Solution of ODEs

The LT can be useful in the solution of linear ODEs with certain non-homogeneous terms (right-hand side function $f(t)$). By applying the LT on both sides of the ODE, it is recast as an algebraic equation, which may be easier to solve. Then, one applies the ILT to recover the original dependent variable.

As a general example, consider the following second-order linear ODE:

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + cy = f(t) \tag{25}$$

with given values of $y(0)$ and $y'(0)$. From a table of LTs, we know that:

$$\begin{aligned} \mathcal{L}\{y''(t)\} &= s^2 Y(s) - sy(0) - y'(0) \\ \mathcal{L}\{y'(t)\} &= sY(s) - y(0) \end{aligned}$$

Therefore, by applying the LT to both sides of the original ODE in equation (25) and using the expressions above, the original ODE becomes an algebraic equation in terms of $Y(s)$,

$$a[s^2 Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$$

and solving for $Y(s)$ yields

$$Y(s) = \left[\frac{y(0)(as+b) + ay'(0)}{as^2 + bs + c}\right] + \frac{F(s)}{as^2 + bs + c} \tag{26}$$

By applying the ILT to both sides of the algebraic equation above, the general solution to the original ODE in equation (25) is obtained:

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{y(0)(as+b) + ay'(0)}{as^2 + bs + c}\right\}}_{\text{Complementary Solution}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{F(s)}{as^2 + bs + c}\right\}}_{\text{Particular Solution}} \tag{27}$$

3.3 MATLAB Functions

The MATLAB functions `laplace` and `ilaplace` compute the LT and ILT of a symbolic function variable, respectively.

4 Partial Differential Equations: Finite Difference Method

Partial Differential Equations (PDEs) can be numerically solved using different methods, such as Finite Difference Method (FDM), Finite Element Method (FEM), and Finite Volume Method (FVM). These methods are characterized by discretizing PDEs and predicting the dependent variables at grid points. This is a very similar strategy to the one used in the numerical methods for ODEs.

For ODEs, there is only one independent variable. Therefore, the grid is one-dimensional (1-D), and the dependent variable receives only one index. For PDEs, there are at least two independent variables. Thus, the domain is a mesh grid, and the dependent variable receives as many indices as there are independent variables. For example, for a PDE in terms of $u(x, y)$, we may use the indices i and j to represent the grid points for the independent variables x and y, respectively. Thus, the discretized dependent variable is expressed as $u_{i,j}$. Notice that we now have a matrix (2-D) of values for the dependent variable. Each value of $u_{i,j}$ is computed based at each pair of points (x_i, y_j).

Consider the 2-D PDE (Laplace equation) defined over a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{28}$$

and boundary conditions

$$u(x, y = y_a) = u_0 \tag{29}$$
$$u(x = x_a, y) = u_1 \tag{30}$$
$$u(x, y = y_b) = u_2 \tag{31}$$
$$u(x = x_b, y) = u_3 \tag{32}$$

The geometry of the problem is depicted in Figure 3.

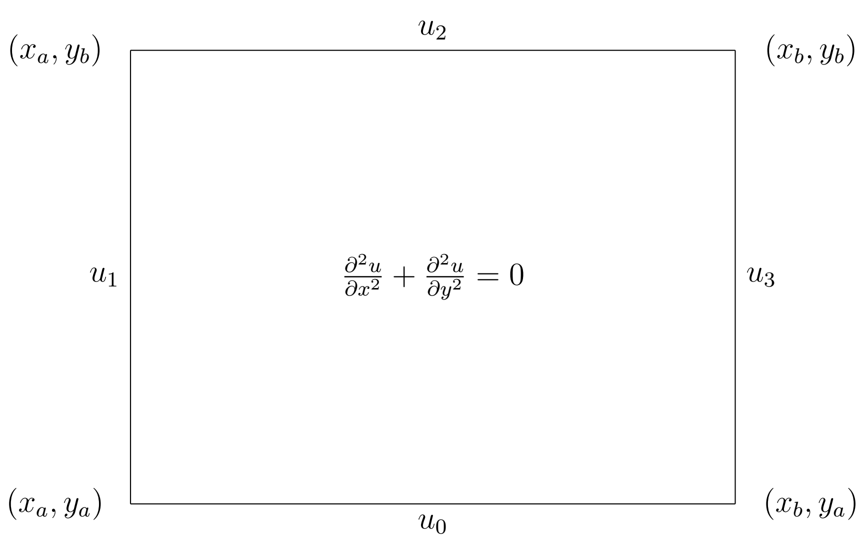

Figure 3: Geometry of the Laplace equation in 2-D.

In the FDM, we consider fixed grids in both x and y dimensions. Therefore, $x_i = x_{i-1} + \Delta x$, for $i = 1, \ldots, M$, and $y_j = y_{j-1} + \Delta y$, for $j = 1, \ldots, N$. Also, $\Delta x = \frac{x_b - x_a}{M}$ and $\Delta y = \frac{y_b - y_a}{N}$. Note that we may consider different number of grid points in each dimension, i.e., Δx does not have to be equal to Δy.

We can use, for example, central finite differences to approximate the derivatives:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \tag{33}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \tag{34}$$

Substituting expressions in equations (33) and (34) into equation (28) gives

$$\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}\right) + \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}\right) = 0, \quad i = 1, \ldots, M-1,\ j = 1, \ldots, N-1$$

which can be rearranged (isolating $u_{i,j}$) as follows

$$u_{i,j} = \frac{1}{2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)} \left[\left(\frac{u_{i+1,j} + u_{i-1,j}}{\Delta x^2}\right) + \left(\frac{u_{i,j+1} + u_{i,j-1}}{\Delta y^2}\right)\right],$$

$$i = 1, \ldots, M-1,\ j = 1, \ldots, N-1 \tag{35}$$

Note that equation (35) is only valid at *inner* grid points. The points on the boundaries of the domain are given by the boundary conditions, which can be rewritten as follows:

$$u_{0,j} = u_0, \quad j = 0, \ldots, N \tag{36}$$

$$u_{i,0} = u_1, \quad i = 1, \ldots, M \tag{37}$$

$$u_{M,j} = u_2, \quad j = 1, \ldots, N \tag{38}$$

$$u_{i,N} = u_3, \quad i = 1, \ldots, M-1 \tag{39}$$

A computer program (such as MATLAB) can be used to calculate the values of $u_{i,j}$. First, we set the boundary values of $u_{i,j}$ with simple assignment statements. Then we initialize the values of $u_{i,j}$ at inner grid points (for example using the average of the boundary values of u). Finally, the values of $u_{i,j}$ at inner grid points are calculated using equation (35) and updated after a number of iterations until some convergence criterion is met.

4.1 Numerical Example

Consider the rectangle shown in Figure 3. Let $x_a = 0$, $x_b = 4$, $y_a = 0$, $y_b = 3$, $u_0 = 100$, $u_1 = 200$, $u_2 = 300$, and $u_3 = 200$. The x dimension is discretized with $M = 50$ and the y dimension is discretized with $N = 40$. The maximum number of iterations allowed is $5,000$, and the convergence tolerance is 10^{-6} based on the Frobenius norm (similar to the 2-norm for vectors) of the u matrix at the current iteration and at the previous iteration. See the accompanying MATLAB script M-File `LaplaceEquationFDM.m` for the code.

Figure 4 shows the error (Frobenius norm) at each iteration. Figure 5 shows the u profile in the xy plane.

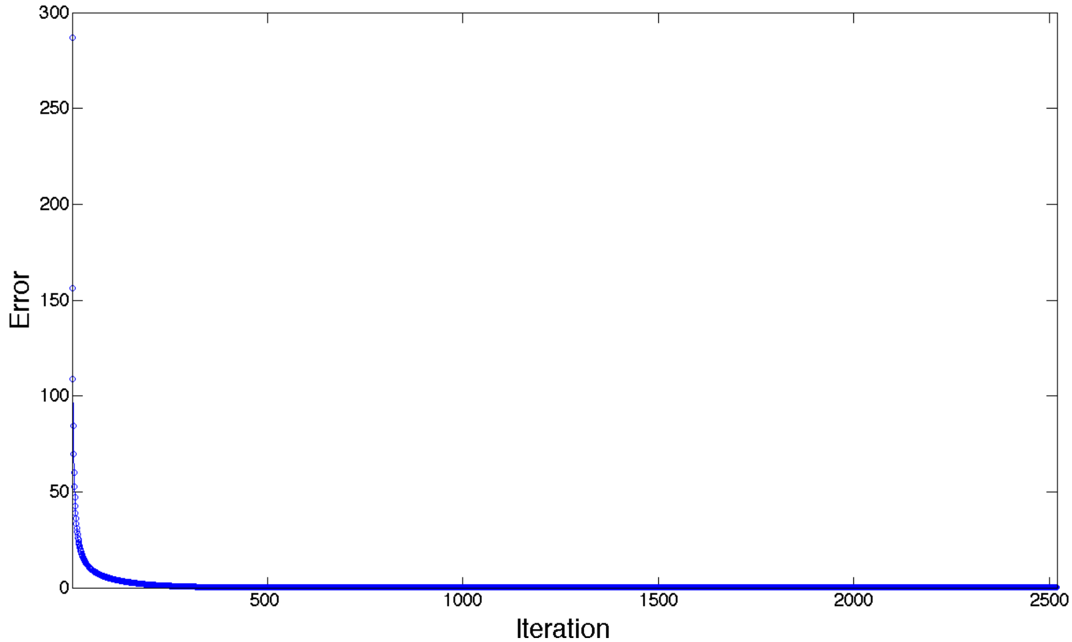

Figure 4: Iteration error (Frobenius norm) for solving the Laplace equation on a rectangle.

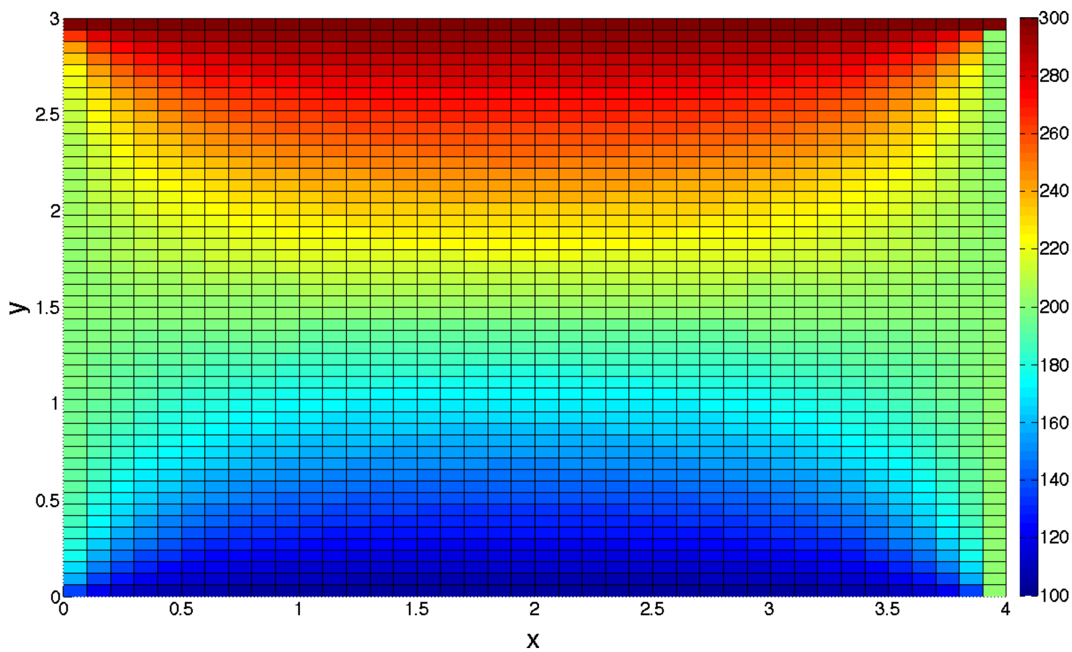

Figure 5: Profile of the solution to the Laplace equation on a rectangle.

4.2 MATLAB Functions

In MATLAB, you can use the PDE Toolbox to define geometries and solve different types of PDEs. Some functions are available depending on the type of the PDE: `pdepe`, `hyperbolic`, `parabolic`, `pdenonlin`.

5 MATLAB Cheat Table

Consult the documentation (Help) for further information and specific examples.

MATLAB Function	Description	Basic Usage
`det`	Determinant of a matrix	`det(A)`
`lu`	LU decomposition	`[L,U] = lu(A)`
`\`	Gaussian elimination	`x = A\b`
`eig`	Eigenvalues and eigenvectors	`[V,D] = eig(A)`
`syms`	Creates symbolic variables and functions	`syms a x v(t)`
`pretty`	Displays symbolic expressions prettily	`pretty(x/a)`
`dsolve`	Solves (system of) ODEs analytically	`dsolve(eqn,conds)`
`ode45`	Solves (system of) IVPs numerically	`ode45(fcn,tspan,y0)`
`bvp4c`	Solves (system of) BVPs numerically	`bvp4c(fcn,bcs,solinit)`
`laplace`	Symbolic Laplace transform	`laplace(f,t,s)`
`ilaplace`	Symbolic Inverse Laplace transform	`ilaplace(F,s,t)`
`residue`	Partial fraction expansion	`[r,p,k] = residue(b,a)`

6 Examples

6.1 Linear Algebra

Given a matrix

$$A = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where α and β are non-zero real numbers, find its eigenvalues and eigenvectors. Find the respective conditions for (a) the eigenvalues to be real and (b) the eigenvectors to be orthogonal.

Answer:

The eigenvalues λ and eigenvectors v are scalars and vectors, respectively, such that

$$A\lambda = \lambda v$$

or

$$(A - \lambda I)v = 0$$

To compute the eigenvalues, find the roots of the characteristic polynomial as follows:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & \alpha & 0 \\ \beta & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

or

$$(1 - \lambda)\left[(1 - \lambda)^2 - \alpha\beta\right] = 0$$

Clearly, one eigenvalue is $\lambda_1 = 1$. The other two eigenvalues are

$$\begin{aligned} (1 - \lambda)^2 - \alpha\beta &= 0 \\ (1 - \lambda)^2 &= \alpha\beta \\ 1 - \lambda &= \pm\sqrt{\alpha\beta} \end{aligned}$$

thus, $\lambda_2 = 1 - \sqrt{\alpha\beta}$ and $\lambda_3 = 1 + \sqrt{\alpha\beta}$.

The eigenvectors are obtained as follows. The eigenvector v_1 associated with λ_1 is given by:

$$\begin{bmatrix} 1-1 & \alpha & 0 \\ \beta & 1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\alpha v_{1,2} = 0$$
$$\beta v_{1,1} = 0$$

Therefore, $v_{1,1} = v_{1,2} = 0$ and $v_{1,3} = k$ (can make $k = 1$). The eigenvector v_2 associated with λ_2 is given by:

$$\begin{bmatrix} 1-1+\sqrt{\alpha\beta} & \alpha & 0 \\ \beta & 1-1+\sqrt{\alpha\beta} & 0 \\ 0 & 0 & 1-1+\sqrt{\alpha\beta} \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{\alpha\beta} & \alpha & 0 \\ \beta & \sqrt{\alpha\beta} & 0 \\ 0 & 0 & \sqrt{\alpha\beta} \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\sqrt{\alpha\beta} v_{2,1} + \alpha v_{2,2} = 0$$
$$\beta v_{2,1} + \sqrt{\alpha\beta} v_{2,2} = 0$$
$$\sqrt{\alpha\beta} v_{2,3} = 0$$

Therefore, $v_{2,2} = k$, $v_{2,1} = \frac{\alpha}{\sqrt{\alpha\beta}} k$ (can make $k = 1$), and $v_{2,3} = 0$. Lastly, the eigenvector v_3 associated with λ_3 is given by:

$$\begin{bmatrix} 1-1-\sqrt{\alpha\beta} & \alpha & 0 \\ \beta & 1-1-\sqrt{\alpha\beta} & 0 \\ 0 & 0 & 1-1-\sqrt{\alpha\beta} \end{bmatrix} \begin{bmatrix} v_{3,1} \\ v_{3,2} \\ v_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\sqrt{\alpha\beta} & \alpha & 0 \\ \beta & -\sqrt{\alpha\beta} & 0 \\ 0 & 0 & -\sqrt{\alpha\beta} \end{bmatrix} \begin{bmatrix} v_{3,1} \\ v_{3,2} \\ v_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$
\begin{aligned}
-\sqrt{\alpha\beta}v_{3,1} + \alpha v_{3,2} &= 0 \\
\beta v_{3,1} - \sqrt{\alpha\beta}v_{3,2} &= 0 \\
-\sqrt{\alpha\beta}v_{3,3} &= 0
\end{aligned}
$$

Therefore, $v_{3,2} = k$, $v_{3,1} = -\frac{\alpha}{\sqrt{\alpha\beta}}k$ (can make $k = 1$), and $v_{3,3} = 0$.

In summary, the eigensystem of A is given by:

$$
\boxed{D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\sqrt{\alpha\beta} & 0 \\ 0 & 0 & 1+\sqrt{\alpha\beta} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 0 & \frac{\alpha}{\sqrt{\alpha\beta}} & -\frac{\alpha}{\sqrt{\alpha\beta}} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}
$$

The eigenvalues are real if $\boxed{\alpha\beta > 0}$, and the eigenvectors are orthogonal if $v_i \cdot v_j = 0$ for $i \neq j$. Clearly, $v_1 \cdot v_2 = v_1 \cdot v_3 = 0$. We must also have that $v_2 \cdot v_3 = -\frac{\alpha^2}{\alpha\beta} + 1 = -\frac{\alpha}{\beta} + 1 = 0$; thus, $\boxed{\alpha = \beta}$.

6.2 First-Order ODE: Integrating Factor

Solve the differential equation

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cos x = 1$$

subject to the boundary condition $y(\pi/2) = 1$.

Answer:

Rearrange the ODE as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\cos x}{\sin x}y = \frac{1}{\sin x}$$

which becomes of the form of equation (7). The integrating factor is:

$$
\begin{aligned}
\mu(x) &= \exp\left(\int 2\frac{\cos x}{\sin x}\mathrm{d}x\right) \\
\mu(x) &= \exp\left(2\ln|\sin x|\right) \\
\mu(x) &= \exp\left(\ln|\sin x|^2\right) \\
\mu(x) &= \sin^2 x
\end{aligned}
$$

Therefore, the general solution is:

$$
\begin{aligned}
y(x) &= \frac{1}{\sin^2 x}\left[\int \frac{\sin^{\cancel{2}} x}{\cancel{\sin x}}\mathrm{d}x + C\right] \\
y(x) &= \frac{1}{\sin^2 x}\left[\int \sin x\mathrm{d}x + C\right]
\end{aligned}
$$

$$\boxed{y(x) = \frac{1}{\sin^2 x}\left[-\cos x + C\right]}$$

6.3 Second-Order ODE: Analytical and Numerical Solution

Find the analytical general solution of

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x$$

given that $y(1) = 1$ and $y(e) = 2e$. Also, use central finite differences to convert the ODE into a tridiagonal matrix system of equations.

Answer:

The general solution of this linear ODE has two parts:

$$y(x) = y_c(x) + y_p(x)$$

Begin with the homogeneous ODE associated with the original ODE:

$$x^2\frac{\mathrm{d}^2y_c}{\mathrm{d}x^2} - x\frac{\mathrm{d}y_c}{\mathrm{d}x} + y_c = 0$$

This is an Euler-Cauchy equation. By assuming the solution to the homogeneous ODE associated with the original ODE of the form $y_c(x) = x^\lambda$, the characteristic equation is given by:

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots are $\lambda_1 = \lambda_2 = \lambda = 1$. Since there is one repeated root, the complementary solution is

$$y_c(x) = C_1 x + C_2 x \ln x$$

Even though the non the nonhomogeneous term (right-hand side of the original ODE) is a polynomial and the ODE is linear, the coefficients of the derivatives of $y(x)$ and itself are *not* constants. Therefore, we cannot *directly* apply the Method of Undetermined Coefficients. However, the RHS term makes this a special case of the Euler-Cauchy equation. For more information, see **Theorem 1**, result (ii) in article De Leon, D. (2010). *Euler-Cauchy Using Undetermined Coefficients*. The College Mathematics Journal, 41(3), 235-237, URL: http://www.jstor.org/stable/10.4169/074683410X488728. According to that theorem, we set $y_p(x) = (Ax + B)\ln^2 x$ (see the *Optional* part at the end of the solution of this problem for the complete steps without making use of that theorem). Substituting the trial $y_p(x)$ into the original ODE, simplifying, and matching coefficients,

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}[(Ax+B)\ln^2 x] - x\frac{\mathrm{d}}{\mathrm{d}x}[(Ax+B)\ln^2 x] + [(Ax+B)\ln^2 x] = x$$
$$2Ax + B(2 - 4\ln x + \ln^2 x) = x$$

Thus, $2A = 1 \therefore A = \frac{1}{2}$ and $B = 0$. The general solution is then

$$y(x) = C_1 x + C_2 x \ln x + \frac{1}{2}x\ln^2 x$$

Applying the boundary conditions yields

$$y(1) = C_1(1) + C_2(1)\ln 1 + \frac{1}{2}(1)\ln^2 1 = 1$$
$$y(e) = C_1(e) + C_2(e)\ln e + \frac{1}{2}(e)\ln^2 e = 2e$$

or

$$C_1 = 1$$
$$eC_1 + eC_2 + \frac{1}{2}e = 2e$$

Therefore, $C_1 = 1$ and $C_2 = \frac{1}{2}$. The analytical general solution is

$$\boxed{y(x) = x + \frac{1}{2}x\ln x + \frac{1}{2}x\ln^2 x}$$

The tridiagonal matrix system of equations arises from approximating the derivatives with finite differences and discretizing the domain. The boundary conditions are represented by the equations:

$$\boxed{\begin{aligned} y_0 &= 1 \\ y_{N+1} &= 2e \end{aligned}}$$

The equations evaluated at inner grid points are obtained by applying central finite difference formulas for the derivatives, which are given by:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} \approx \frac{y_{n+1} - y_{n-1}}{2h}$$

Substituting the above expressions into the original ODE yields

$$x_n^2\left(\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}\right) - x_n\left(\frac{y_{n+1} - y_{n-1}}{2h}\right) + y_n = x_n, \quad n = 1, \ldots, N$$

or

$$\boxed{[x_n(x_n - 0.5h)]\, y_{n+1} + (h^2 - 2x_n^2)y_n + [x_n(x_n + 0.5h)]\, y_{n-1} = h^2 x_n, \quad n = 1, \ldots, N}$$

where $x_n = x_{n-1} + h = 1 + nh$, for $n = 1, \ldots, N+1$, and $h = \frac{e-1}{N}$.

Optional: Here are all the steps to obtain the analytical solution without making use of the aforementioned theorem.

The trick is to convert the Euler-Cauchy equation, which does not have constant coefficients, into an equivalent ODE that has constant coefficients and a suitable RHS so that the Method of Undetermined Coefficients can be used to solve the transformed ODE. The transformation is in the independent variable: $x = e^s$. Here, s is a new independent variable, and we can also write $s = \ln x$. Now, we have to convert all the derivatives with respect to x to derivatives with respect to s using the chain and product rules. Start with the first-order derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s}\frac{1}{x} \tag{40}$$

Use the above to convert the second-order derivative with respect to x:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\frac{1}{x}\right)$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right) - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{1}{x^2}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right) - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{1}{x^2}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x}\frac{\mathrm{d}^2y}{\mathrm{d}s^2}\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right) - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{1}{x^2}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{1}{x^2} \tag{41}$$

Plug the expressions in equations (40) and (41) into the original Euler-Cauchy equation and the resulting ODE has s as the independent variable:

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - 2\frac{\mathrm{d}y}{\mathrm{d}s} + y = e^s$$

with boundary conditions $y(0) = 1$ and $y(1) = 2e$. Note that the new ODE has constant coefficients and a suitable RHS term. The complementary solution is (repeated root $\lambda = 1$):

$$y_c(s) = (C_1 + C_2 s)e^s$$

Now, we can directly apply the Method of Undetermined Coefficients. Since the RHS term is present in $y_c(s)$, we have to multiply it by powers of s until it does not contain terms in common with $y_c(s)$. Thus, the trial particular solution is $y_p(s) = Ae^{Bs}s^2$. After plugging $y_p(s)$ into the new ODE and matching terms, we obtain $A = \frac{1}{2}$ and $B = 1$. Therefore, the general solution is:

$$y(s) = (C_1 + C_2 s)e^s + \frac{1}{2}e^s s^2$$

After applying the new boundary conditions, we obtain $C_1 = 1$ and $C_2 = \frac{1}{2}$. Finally, we recover the original independent variable x (recall that $s = \ln x$) and write the final solution for the original Euler-Cauchy equation:

$$y(x) = (C_1 + C_2 \ln x)x + \frac{1}{2}x\ln^2 x$$

6.4 System of ODEs: Laplace Transform

Use Laplace transforms to solve, for $t \geq 0$, the differential equations

$$\ddot{x} + 2x + y = \cos t$$
$$\ddot{y} + 2x + 3y = 2\cos t$$

which describe a coupled system that starts from rest at the equilibrium position.

Answer:

From the problem statement, $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$. Applying the Laplace transform to both sides of the ODEs gives

$$s^2X(s) + 2X(s) + Y(s) = \frac{s}{s^2+1}$$
$$s^2Y(x) + 2X(s) + 3Y(s) = 2\frac{s}{s^2+1}$$

or

$$(s^2+2)X(s) + Y(s) = \frac{s}{s^2+1}$$
$$2X(s) + (s^2+3)Y(s) = 2\frac{s}{s^2+1}$$

Solving the linear algebraic system above for $X(s)$ and $Y(s)$ via substitution yields

$$X(s) = \frac{s}{(s^2+1)(s^2+4)}$$
$$Y(s) = 2\frac{s}{(s^2+1)(s^2+4)}$$

Expanding the fractions in partial fractions gives

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$
$$Y(s) = \frac{2}{3}\frac{s}{s^2+1} - \frac{2}{3}\frac{s}{s^2+4}$$

Applying the inverse Laplace transform on both sides of the equations above gives the final solution

$$\boxed{\begin{aligned} x(t) &= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t \\ y(t) &= \frac{2}{3}\cos t - \frac{2}{3}\cos 2t \end{aligned}}$$

Note that a plot of $y(t)$ vs. $x(t)$ is a line in the xy-plane, since $y = 2x$.